# Neutrino Phenomenology

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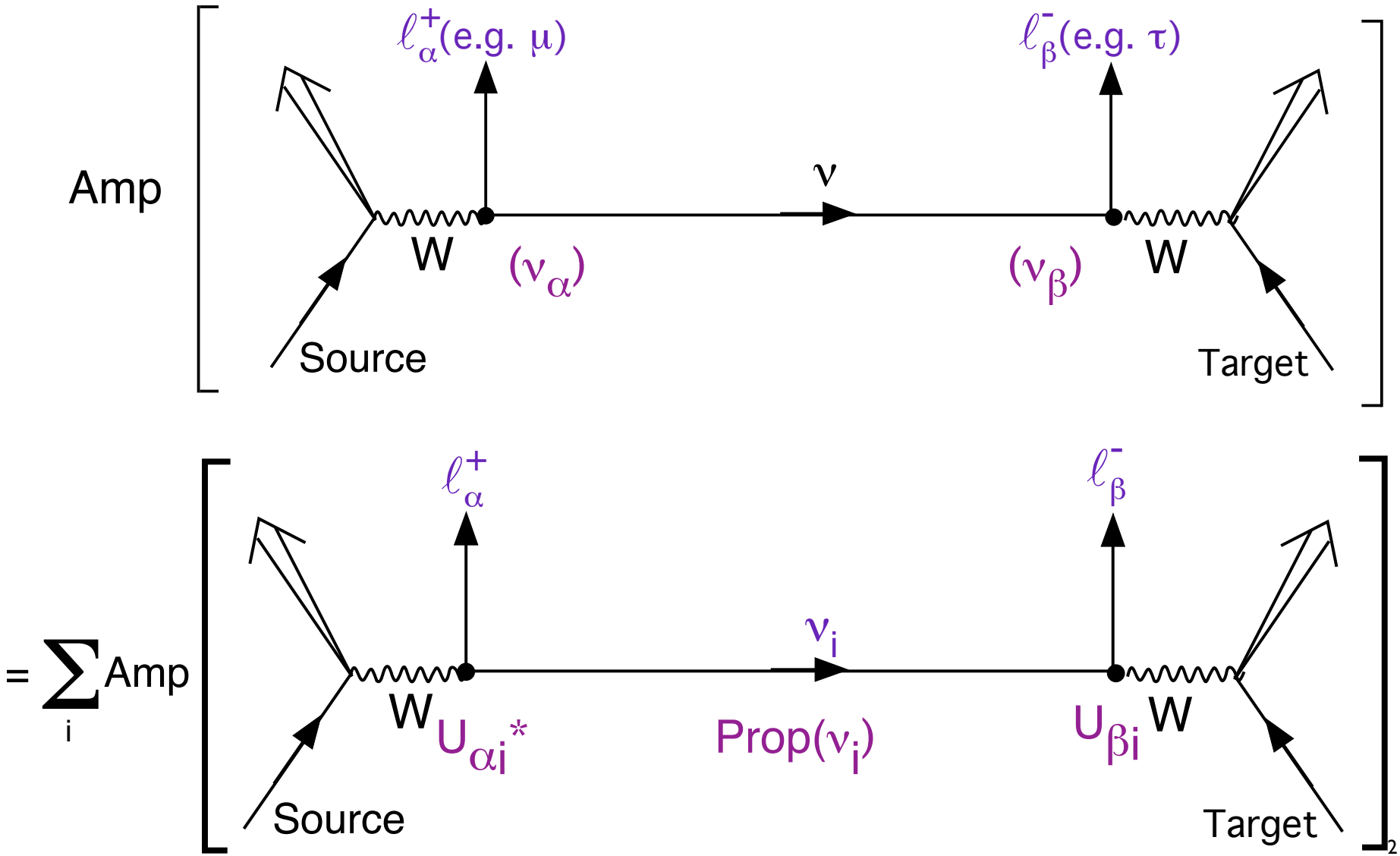
Part 2

# Neutrino Flavor Change (Oscillation) in Vacuum

(Approach of B.K. & Stodolsky)



$$\text{Amp}\,[\nu_\alpha \rightarrow \nu_\beta] = \Sigma U_{\alpha i}{}^{*}\,\text{Prop}(\nu_i) U_{\beta i}$$

What is Propagator $(\nu_i) \equiv \text{Prop}(\nu_i)$?

In the $\nu_i$ rest frame, where the proper time is $\tau_i$,

$$i\frac{\partial}{\partial \tau_i}|\nu_i(\tau_i)>= m_i|\nu_i(\tau_i)> \quad .$$

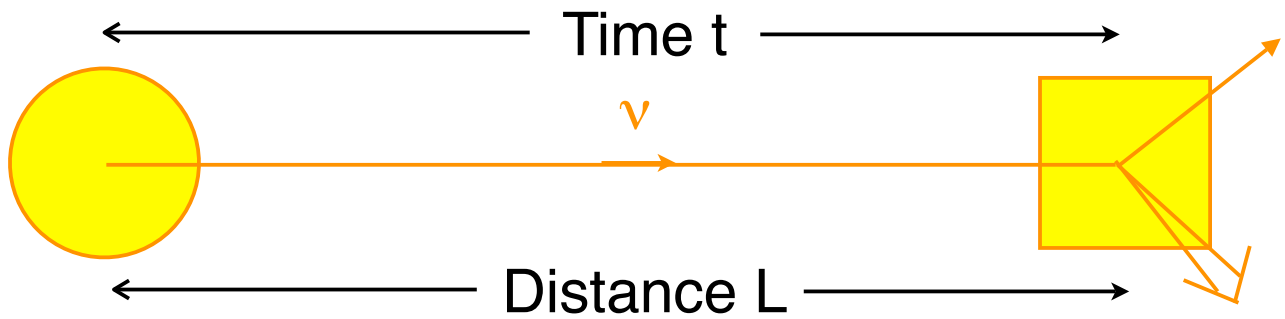
Thus,

$$|\nu_i(\tau_i)>= e^{-im_i\tau_i}|\nu_i(0)> \quad .$$

Then, the amplitude for propagation for time $\tau_i$ is —

$$\text{Prop}(\nu_i) \equiv < \nu_i(0)|\nu_i(\tau_i)>= e^{-im_i\tau_i} \quad .$$

In the laboratory frame —




The experimenter chooses L and t.

They are common to all components of the beam.

For each $\nu_i$, by Lorentz invariance,

$$(E_i, p_i) \times (t, L) = m_i\tau_i = E_i t - p_i L \ .$$

Neutrino sources are ~ constant in time.

Averaged over time, the

$$e^{-iE_1t} - e^{-iE_2t}$$ interference

is —

$$< e^{-i(E_1-E_2)t} >_t = 0$$

unless $E_2 = E_1$ .

Only neutrino mass eigenstates with a common energy E are coherent.

(Stodolsky)

For each mass eigenstate ,

$$p_i = \sqrt{E^2 - m_i^2} \cong E - \frac{m_i^2}{2E} \quad .$$

Then the phase in the $\nu_i$ propagator $exp[-im_i\tau_i]$ is —

$$m_i\tau_i = E_i t - p_i L \cong Et - (E - m_i^2/2E)L$$

$$= \underbrace{E(t - L)}_{\text{Irrelevant overall phase}} + m_i^2 L/2E \quad .$$

What if the neutrino source is ***not*** constant in time?

The relative phase between two mass eigenstates,

$$\delta\phi(21) \equiv (E_2 t - p_2 L) - (E_1 t - p_1 L) \quad ,$$

is unchanged.

(Lipkin)

An approximation to the average speed of the $\nu_1$ and $\nu_2$ waves is

$$\bar{v} \equiv \frac{p_1 + p_2}{E_1 + E_2} \quad .$$

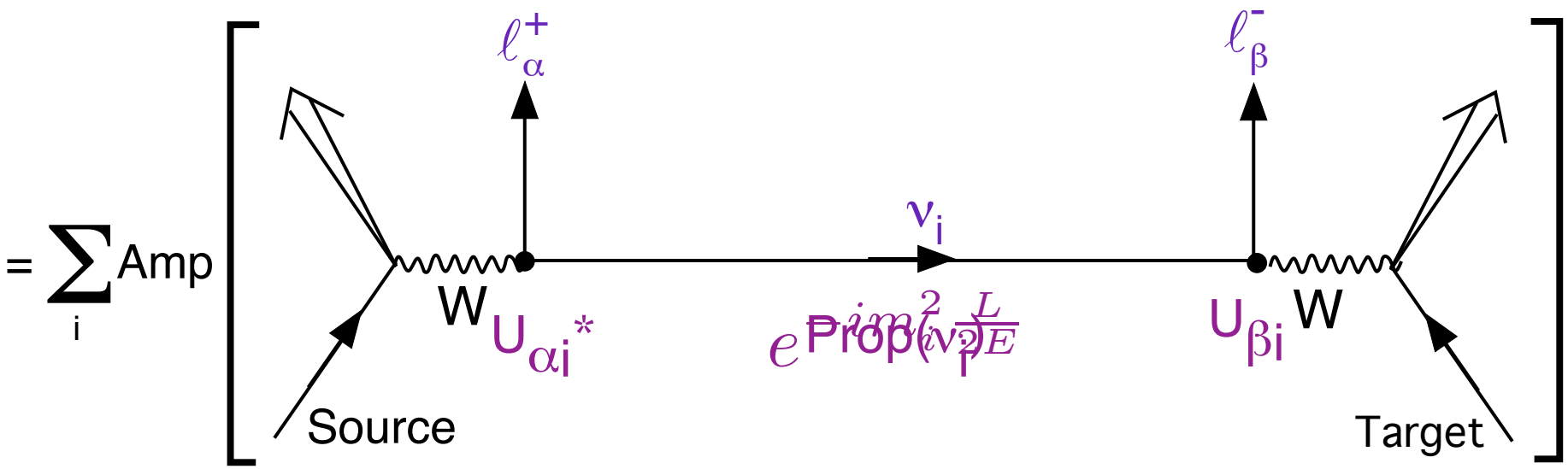
Then the travel time $t \cong L/\bar{v}$ .

Thus,

$$\delta\phi(21) = (p_1 - p_2)L - (E_1 - E_2)t$$

$$\cong \frac{p_1^2 - p_2^2}{p_1 + p_2} L - \frac{E_1^2 - E_2^2}{p_1 + p_2} L \cong (m_2^2 - m_1^2)L/2E$$

$$\mathrm{Amp}\,[\nu_\alpha \to \nu_\beta]$$




$$= \sum_i U^*_{\alpha i} e^{-i m_i^2 \frac{L}{2E}} U_{\beta i}$$

# Probability for Neutrino Oscillation in Vacuum

$$P(\nu_\alpha \to \nu_\beta) = |\mathrm{Amp}(\nu_\alpha \to \nu_\beta)|^2 =$$

$$= \delta_{\alpha\beta} - 4\sum_{i>j} \Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2(\Delta m^2_{ij} \frac{L}{4E})$$

$$+2\sum_{i>j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin(\Delta m^2_{ij} \frac{L}{2E})$$

where $\Delta m^2_{ij} \equiv m^2_i - m^2_j$

# For Antineutrinos –

We assume the world is CPT invariant.

Our formalism assumes this.

$$P(\overline{\nu_\alpha} \to \overline{\nu_\beta}) \overset{CPT}{=} P(\nu_\beta \to \nu_\alpha) = P(\nu_\alpha \to \nu_\beta; U \to U^*)$$

Thus,

$$P(\overset{(-)}{\nu_\alpha} \to \overset{(-)}{\nu_\beta}) =$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} \Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2(\Delta m^2_{ij} \frac{L}{4E})$$

$$\underset{(-)}{+} 2 \sum_{i>j} \Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin(\Delta m^2_{ij} \frac{L}{2E})$$

A complex U would lead to the CP violation

$P(\overline{\nu_\alpha} \to \overline{\nu_\beta}) \neq P(\nu_\alpha \to \nu_\beta)$ .

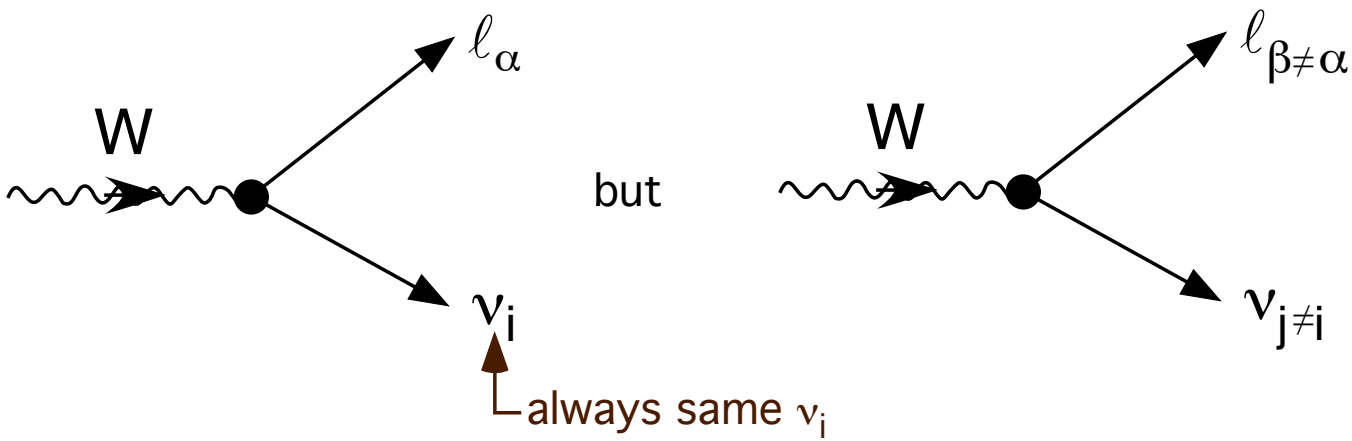
# — Comments —

1. If all $m_i = 0$, so that all $\Delta m_{ij}^2 = 0$,

$$P\left(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta\right) = \delta_{\alpha\beta}$$

Flavor *change* $\Rightarrow$ $\nu$ Mass

2. If there is no mixing,




$$\Rightarrow U_{\alpha i} U_{\beta\neq\alpha,i} = 0, \text{ so that } P\left(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta\right) = \delta_{\alpha\beta}.$$

Flavor *change* $\Rightarrow$ Mixing

3. One can detect ($\nu_\alpha \to \nu_\beta$) in two ways:

See $\nu_{\beta\neq\alpha}$ in a $\nu_\alpha$ beam (Appearance)

See some of known $\nu_\alpha$ flux disappear (Disappearance)

4. Including $\hbar$ and c

$$\Delta m^2 \frac{L}{4E} = 1.27\Delta m^2(\mathrm{eV}^2)\frac{L(\mathrm{km})}{E(\mathrm{GeV})}$$

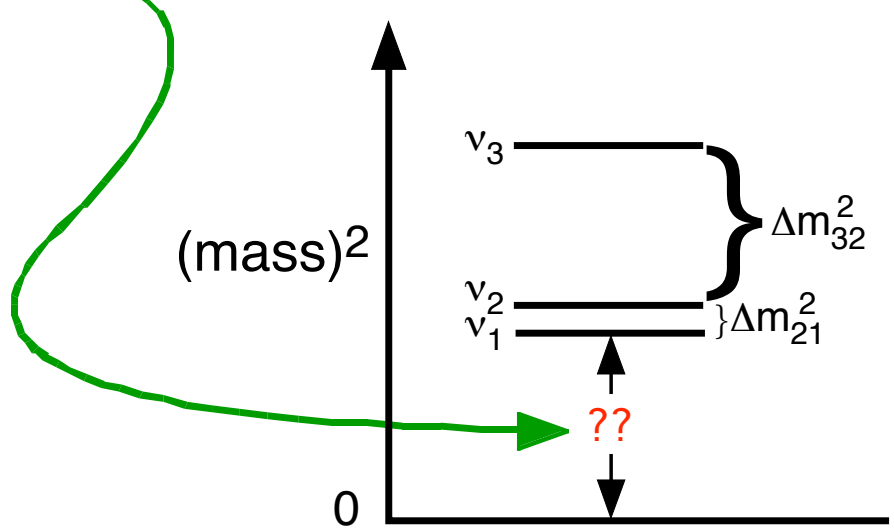
$\sin^2[1.27\Delta m^2(\mathrm{eV})^2 \frac{L(\mathrm{km})}{E(\mathrm{GeV})}]$ becomes appreciable when its argument reaches $\mathcal{O}(1)$.

An experiment with given L/E is sensitive to

$$\Delta m^2(\mathrm{eV}^2) \gtrsim \frac{E(\mathrm{GeV})}{L(\mathrm{km})} \quad .$$

5. Flavor change in vacuum oscillates with L/E. Hence the name "neutrino oscillation". {The L/E is from the proper time τ.}

6. P ($\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta$) depends only on squared-mass splittings. Oscillation experiments cannot tell us



7. Neutrino flavor change does not change the total flux in a beam.

It just redistributes it among the flavors.

$$\sum_{\text{All } \beta} P(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) = 1$$

But some of the flavors β≠α could be sterile.

Then some of the *active* flux disappears:

$$\phi_{\nu_e} + \phi_{\nu_\mu} + \phi_{\nu_\tau} < \phi_{\text{Original}}$$

8. Assuming all coherent $\nu_i$ in a beam have a common momentum p, rather than a common energy E, is a harmless error.

This assumption leads to the same $P(\overset{(-)}{\nu}_\alpha \rightarrow \overset{(-)}{\nu}_\beta)$.

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# Important Special Cases

## Three Flavors

For $\beta \neq \alpha$,

$$e^{-im_1^2 \frac{L}{2E}} \mathrm{Amp}^*(\nu_\alpha \to \nu_\beta) = \sum_i U_{\alpha i} U^*_{\beta i} e^{im_i^2 \frac{L}{2E}} e^{-im_1^2 \frac{L}{2E}}$$

$$= U_{\alpha 3} U^*_{\beta 3} e^{2i\Delta_{31}} + U_{\alpha 2} U^*_{\beta 2} e^{2i\Delta_{21}} \underbrace{-(U_{\alpha 3} U^*_{\beta 3} + U_{\alpha 2} U^*_{\beta 2})}_{\text{Unitarity}}$$

$$= 2i[U_{\alpha 3} U^*_{\beta 3} e^{i\Delta_{31}} \sin \Delta_{31} + U_{\alpha 2} U^*_{\beta 2} e^{i\Delta_{21}} \sin \Delta_{21}]$$

$$\text{where } \Delta_{ij} \equiv \Delta m^2_{ij} \frac{L}{4E} \equiv (m_i^2 - m_j^2) \frac{L}{4E} \quad .$$

$$P(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) = \left| e^{-im_1^2 \frac{L}{2E}} \mathrm{Amp}^* (\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) \right|^2$$

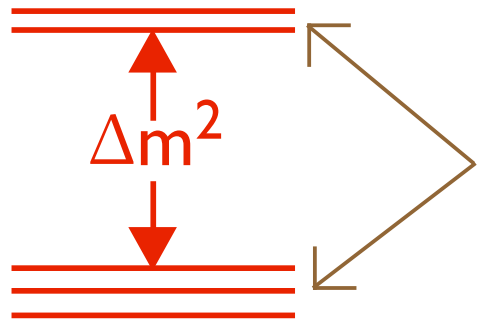
$$= 4[|U_{\alpha 3} U_{\beta 3}|^2 \sin^2 \Delta_{31} + |U_{\alpha 2} U_{\beta 2}|^2 \sin^2 \Delta_{21}$$

$$+2|U_{\alpha 3} U_{\beta 3} U_{\alpha 2} U_{\beta 2}| \sin \Delta_{31} \sin \Delta_{21} \cos(\Delta_{32} \underset{(-)}{+} \delta_{32})] \quad .$$

Here $\delta_{32} \equiv \arg(U_{\alpha 3} U^*_{\beta 3} U^*_{\alpha 2} U_{\beta 2})$ , a CP − violating phase.

Two waves of different frequencies,
and their ~~CP~~ interference.

# When One Big $\Delta m^2$ Dominates




These splittings are invisible if $\Delta m^2 \dfrac{L}{E} = \mathcal{O}(1)$.

For $\beta \neq \alpha$,

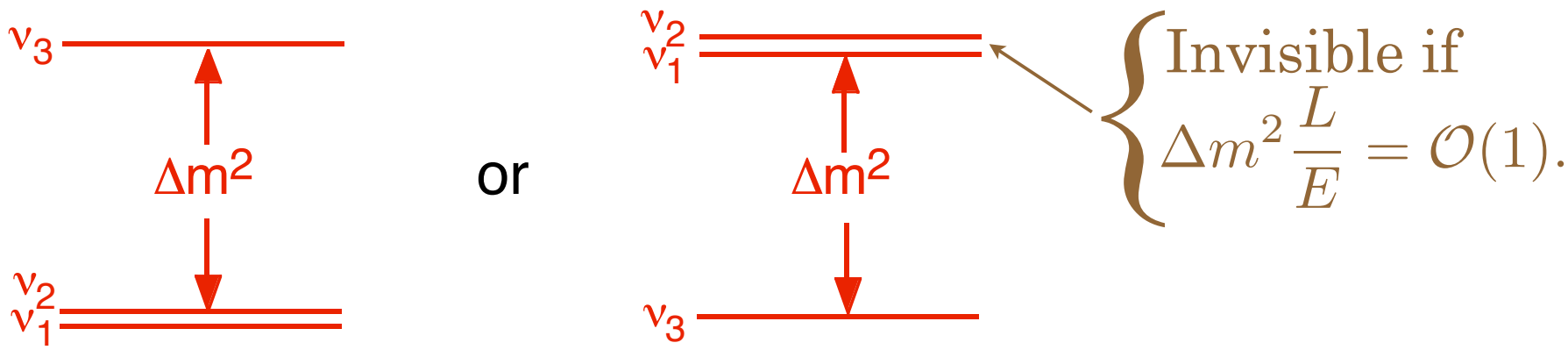
$$P(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) \cong S_{\alpha\beta} \sin^2(\Delta m^2 \frac{L}{4E}) \; ; \; S_{\alpha\beta} \equiv 4 \left| \sum_{i \text{ Clump}} U^*_{\alpha i} U_{\beta i} \right|^2 .$$

For no flavor change,

$$P(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\alpha) \cong 1 - 4T_\alpha(1 - T_\alpha) \sin^2(\Delta m^2 \frac{L}{4E}) \; ; \; T_\alpha \equiv \sum_{i \text{ Clump}} |U^*_{\alpha i}|^2 .$$

"i Clump" is a sum over only the mass eigenstates on one end of the big gap $\Delta m^2$.

# When the Spectrum Is—




For $\beta \neq \alpha$,

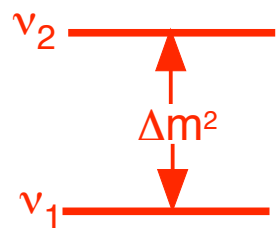
$$\mathrm{P}(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) \cong 4|U_{\alpha 3}U_{\beta 3}|^2 \sin^2(\Delta m^2 \tfrac{L}{4E}) \quad .$$

For no flavor change,

$$\mathrm{P}(\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta) \cong 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2) \sin^2(\Delta m^2 \tfrac{L}{4E}) \quad .$$

Experiments with $\Delta m^2 \frac{L}{E} = \mathcal{O}(1)$ can determine the flavor content of $\nu_3$.

# When There are Only Two Flavors and Two Mass Eigenstates




Majorana
~~CP~~ phase

$$U = \begin{bmatrix} U_{\alpha 1} & U_{\alpha 2} \\ U_{\beta 1} & U_{\beta 2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{i\xi} & 0 \\ 0 & 1 \end{bmatrix}$$

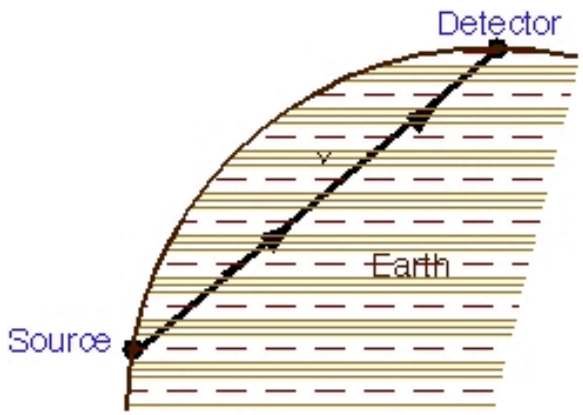
Mixing angle

$$S_{\alpha\beta} = 4T_\alpha(1 - T_\alpha) = \sin^2 2\theta$$

For $\beta \neq \alpha$, $\quad P(\overset{(-)}{\nu_\alpha} \leftrightarrow \overset{(-)}{\nu_\beta}) = \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$ .

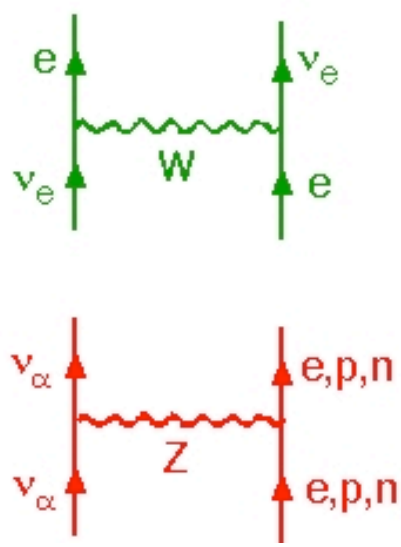
For no flavor change, $P(\overset{(-)}{\nu_\alpha} \rightarrow \overset{(-)}{\nu_\alpha}) = 1 - \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$.

# Neutrino Flavor Change in Matter




Coherent forward scattering from ambient matter can have a big effect.

Interaction




Interaction Potential Energy

$$V_W = +\sqrt{2} G_F N_e \qquad (- \text{ for } \overline{\nu_e})$$

#e/vol

$$V_Z = -\frac{\sqrt{2}}{2} G_F N_n \qquad (+ \text{ for } \overline{\nu_\alpha})$$

#n/vol

Neutrino propagation in matter is conveniently treated via a Schrödinger Equation:

$$i\frac{\partial}{\partial t}\nu(t) = H\,\nu(t)$$

Matrix in flavor space

Multi-component in flavor space

To illustrate, we describe the case —

## When Only Two Neutrinos Count

Amp. to be a $\nu_e$

$$\nu(t) = \begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} \quad ; \quad H = \begin{array}{c} \\ \nu_e \\ \nu_\mu \end{array} \begin{array}{c} \begin{array}{cc} \nu_e & \nu_\mu \end{array} \\ \begin{bmatrix} \sim & \sim \\ \sim & \sim \end{bmatrix} \end{array}$$

Amp. to be a $\nu_\mu$

A 2x2 matrix in $\nu_e$-$\nu_\mu$ space

$$i\frac{\partial}{\partial t}\begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} = \begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix}\begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix}$$

In general

In vacuum

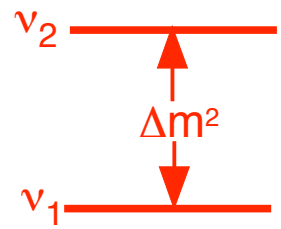
$$< \nu_\alpha|H|\nu_\beta > = < \sum_i U^*_{\alpha i}\nu_i|H|\sum_j U^*_{\beta j}\nu_j > = \sum_j U_{\alpha j}U^*_{\beta j}\sqrt{p^2 + m_j^2}$$

Momentum of the beam

In flavor change, only relative phases, hence relative energies, matter.

$\therefore$ In H, any multiple of the Identity Matrix I may be omitted.

# In Vacuum




$$U = \begin{array}{c c} & \begin{array}{cc} \nu_1 & \nu_2 \end{array} \\ \begin{array}{c} \nu_e \\ \nu_\mu \end{array} & \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{array} \quad ; \quad \begin{array}{l} \nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta \\ \nu_\mu = \nu_1(-\sin\theta) + \nu_2 \cos\theta \end{array}$$

It follows that, omitting a piece ∝ I,

$$H_{\mathrm{Vac}} = \frac{\Delta m^2}{4E} \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} .$$

With Schrödinger's Equation, this gives the usual $P(\nu_e \to \nu_\mu)$.

The eigenvalues of $H_{Vac}$ are —

$$\pm\frac{\Delta m^2}{4E} \equiv \pm\lambda \quad .$$

With $c \equiv \cos\theta$, $s \equiv \sin\theta$,

$$\nu_e = \nu_1 c + \nu_2 s \xrightarrow{t} \nu(t) = \nu_1 c e^{i\lambda t} + \nu_2 s e^{-i\lambda t}$$

$$P(\nu_e \to \nu_\mu) = \ |<\nu_\mu|\nu(t)>|^2 = |sc(-e^{i\lambda t} + e^{-i\lambda t})|^2$$

$$= \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$$

# In Matter

$$H_M = H_{\rm Vac} + V_W \begin{array}{c} \begin{array}{cc} \nu_e & \nu_\mu \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \nu_e \\ \nu_\mu \end{array} + V_Z \underbrace{\begin{array}{c} \begin{array}{cc} \nu_e & \nu_\mu \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \nu_e \\ \nu_\mu \end{array}}_{\propto \text{ I, so drop}}$$

$$H_M = H_{\rm Vac} + \frac{V_W}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{V_W}{2} \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$H_M = \frac{\Delta m^2}{4E} \begin{bmatrix} -(\cos 2\theta - x) & \sin 2\theta \\ \sin 2\theta & \cos 2\theta - x \end{bmatrix} ,$$

with $x \equiv \dfrac{V_W/2}{\Delta m^2/4E} = \dfrac{2\sqrt{2} G_F N_e E}{\Delta m^2}$ .

# The Effective Splitting and Mixing in Matter

If we define —

$$\Delta m_M^2 \equiv \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - x)^2}$$

and

$$\sin^2 2\theta_M \equiv \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \quad ,$$

then

$$H_M = \frac{\Delta m_M^2}{4E} \begin{bmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta_M & \cos 2\theta_M \end{bmatrix} \quad .$$

This is $H_{Vac}$ with $(\Delta m^2, \theta) \rightarrow (\Delta m_M{}^2, \theta_M)$.

Thus, $\Delta m_M{}^2$ and $\theta_M$ are the effective splitting and mixing angle in matter.